

# On integrability of the Yang-Baxter $\sigma$ -model

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## Abstract

We prove the integrability of the Yang-Baxter  $\sigma$ -model which is the Poisson-Lie deformation of the principal chiral model. We find also an explicit one-to-one map transforming every solution of the principal chiral model into a solution of the deformed model. With the help of this map, the standard procedure of the dressing of the principal chiral solutions can be directly transferred into the deformed Yang-Baxter context.

# 1 Introduction

The family of known integrable nonlinear  $\sigma$ -models on group manifolds is not too big. In particular, for every simple compact group target  $G$  there is known the Lax pair of the principal  $\sigma$ -model [12]. Moreover, for the group  $G = SU(2)$ , Cherednik [1] has constructed a Lax pair for certain one-parameter deformation of the principal chiral model usually referred to as the anisotropic principal chiral model. The action of this deformed model reads

$$S_\varepsilon(g) = \frac{1}{1+\varepsilon^2} \int_W (g^{-1}\partial_+g, g^{-1}\partial_-g)_\mathcal{G} + \frac{\varepsilon^2}{1+\varepsilon^2} \int_W ((g^{-1}\partial_+g)_H, (g^{-1}\partial_-g)_H)_\mathcal{G}. \quad (1)$$

Here  $W$  stand for a two-dimensional world-sheet,  $g : W \rightarrow SU(2)$  is a smooth map,  $(\cdot, \cdot)_\mathcal{G}$  is the Killing-Cartan form on the Lie algebra  $\mathcal{G} \equiv Lie(G)$ ,  $\varepsilon$  is the deformation parameter,  $\partial_{\xi^\pm}$  are the standard light-cone derivatives and  $(g^{-1}\partial_\pm g)_H$  are the orthogonal projections of the chiral components of the Maurer-Cartan form on the Cartan subalgebra of  $su(2)$ .

Having studied T-dualizable  $\sigma$ -models, we have recently considered the following  $\varepsilon$ -deformation of the principal chiral model for every simple compact group  $G$  [6]:

$$S_\varepsilon(g) = \int_W (g^{-1}\partial_+g, (I - \varepsilon R)^{-1}g^{-1}\partial_-g)_\mathcal{G}. \quad (2)$$

Here  $I : \mathcal{G} \rightarrow \mathcal{G}$  is the identity map and  $R : \mathcal{G} \rightarrow \mathcal{G}$  is the so called Yang-Baxter operator [10]. In [6], we have called the model (2) the Yang-Baxter  $\sigma$ -model and we have observed therein that, for  $G = SU(2)$ , it coincides with the model (1) considered by Cherednik. The latter observation may seem surprising since the presence of the skew-symmetric Yang-Baxter operator in the Lagrangian should generate a torsion which is not present in the anisotropic model (1). However, it turns out that, for the group  $SU(2)$ , this torsion term is a total derivative and can be omitted.

The fact that, for  $G = SU(2)$ , the Yang-Baxter  $\sigma$ -model becomes the integrable anisotropic principal chiral model makes us conjecture that the model (2) might be integrable for every simple compact group  $G$ . One of the main results of this paper consists in proving this conjecture, i.e. in finding the Lax pair of the Yang-Baxter  $\sigma$ -model. The Cherednik Lax pair of the model (1) was not written in terms of the Yang-Baxter operator  $R$ , hence it cannot be generalized directly to a general target  $G$ . Therefore we have to develop

another method in order to find the Lax pair of the model (2). Anticipating the quantitative analysis presented in the body of the paper, this Lax pair turns out to have the following form

$$A_{\pm}^{\varepsilon}(\lambda) \equiv \left( \varepsilon^2 \mp \varepsilon R - \frac{1 + \varepsilon^2}{1 \pm \lambda} \right) (I \pm \varepsilon R)^{-1} g^{-1} \partial_{\pm} g. \quad (3)$$

Here  $\lambda$  is a complex valued spectral parameter. We note that for  $\varepsilon = 0$  the Yang-Baxter  $\sigma$ -model (2) becomes the principal chiral model and (3) becomes the standard Lax pair introduced by Zakharov and Mikhailov in [12]:

$$A_{\pm}^0(\lambda) = -\frac{g^{-1} \partial_{\pm} g}{1 \pm \lambda}. \quad (4)$$

From a pragmatic point of view, it is not so important how the Lax pair of an integrable model has been constructed (e.g. the "trial and error" method is considered as a honorable one in this context). What is rather important are the consequences of the existence of the Lax pair for the dynamics of the model. We shall indeed discuss these consequences later on (in Section 3.3), but we decided to include in this article also the details of the construction of the Lax pair (3). In fact, the reader will be able to convince himself, that we rather derive the Lax pair (3) than guess it up. The crucial ingredient which makes this derivation possible is the concept of the Poisson-Lie  $\varepsilon$ -deformation which is the classical predecessor of the quantum group  $q$ -deformation (in fact,  $q = e^{\varepsilon}$ ). In particular, we shall work successfully with a hypothesis that the spectral parameter of an integrable model  $A$  can be interpreted as the deformation parameter of the Poisson-Lie deformation  $A_{\varepsilon}$  of the model  $A$ . We do not know whether this hypothesis work for a general integrable model but we give two examples in this paper where it indeed works and it yields very concrete results as it is the Lax pair (3).

The plan of the paper is as follows: in Section 2, we shall expose the background material, in particular, the definition of the Yang-Baxter operator  $R : \mathcal{G} \rightarrow \mathcal{G}$  and the definition of the concept of the Poisson-Lie symmetry of non-linear  $\sigma$ -models. In Section 3, we present original results. In particular, we derive the Lax pair (3) and then integrate it to obtain the so called extended solution (in the sense of Uhlenbeck [11]) of the Yang-Baxter model. We relate this extended solution to the extended solution of the principal chiral model which permits us to construct a one-to-one map  $\Xi_{\varepsilon}$  relating

the ordinary (i.e. non-extended) solutions of the principal chiral model to the ordinary solutions of the Yang-Baxter model. Moreover, we shall succeed to express the map  $\Xi_\epsilon$  explicitly in terms of the group-theoretical Iwasawa map which makes possible to transfer directly the solution generating dressing transformation machinery from the principal chiral context into the deformed Yang-Baxter one. We finish with conclusions and an outlook.

## 2 Background material

### 2.1 Principal chiral model

Consider a simply connected flat world-sheet  $W$ , i.e. a two-dimensional space-time parametrized by the time coordinate  $\tau$  and the space coordinate  $\sigma$ . We introduce also the light-cone coordinates

$$\xi^\pm \equiv \frac{1}{2}(\tau \pm \sigma)$$

and the corresponding light-cone derivatives

$$\partial_\pm = \partial_\tau \pm \partial_\sigma.$$

Let  $G$  be a simple compact connected and simply connected Lie group. We shall call a principal chiral field any smooth map from  $W$  to  $G$ , satisfying an evolution equation

$$\partial_+(g^{-1}\partial_-g) + \partial_-(g^{-1}\partial_+g) = 0 \quad (5)$$

The least action principle which yields the equation (5) reads

$$S = \int_W (g^{-1}\partial_+g, g^{-1}\partial_-g)_G. \quad (6)$$

In what follows we shall concentrate on the dynamics in the bulk and we shall not specify any boundary conditions.

It is well-known [12] that to every solution  $g(\xi^+, \xi^-)$  of the field equations (5), it can be associated a flat  $\mathcal{G}^{\mathbf{C}}$ -valued Lax connection

$$A^0(\lambda) = A_+^0(\lambda)d\xi^+ + A_-^0(\lambda)d\xi^-, \quad (7)$$

where the corresponding Zakharov-Mikhailov Lax pair reads

$$A_{\pm}^0(\lambda) = -\frac{g^{-1}\partial_{\pm}g}{1 \pm \lambda}. \quad (8)$$

This means that the maps  $A_{\pm}^0(\lambda) : W \rightarrow \mathcal{G}^{\mathbf{C}}$  verify the zero-curvature condition in  $\mathcal{G}^{\mathbf{C}}$ , i.e. it holds for every  $\lambda \in \mathbf{C} \setminus \{\pm 1\}$

$$\partial_+ A_-^0(\lambda) - \partial_- A_+^0(\lambda) + [A_-^0(\lambda), A_+^0(\lambda)] = 0. \quad (9)$$

Reciprocally, consider two  $\mathcal{G}$ -valued fields  $u_{\pm}(\xi^+, \xi^-)$  and a  $\mathcal{G}^{\mathbf{C}}$ -valued connection  $A^0(\lambda) = A_+^0(\lambda)d\xi^+ + A_-^0(\lambda)d\xi^-$  where

$$A_{\pm}^0(\lambda) = -\frac{u_{\pm}}{1 \pm \lambda}. \quad (10)$$

The flatness of this connection for every value of the spectral parameter  $\lambda$  implies two equations for  $u_{\pm}$ :

$$\begin{aligned} \partial_+ u_- - \partial_- u_+ - [u_-, u_+] &= 0, \\ \partial_- u_+ + \partial_+ u_- &= 0. \end{aligned} \quad (11)$$

The first of this equations is itself a zero curvature condition in the (compact) Lie algebra  $\mathcal{G}$  which means that there is a  $G$ -valued field  $g(\xi^+, \xi^-)$  such that

$$u_{\pm} = g^{-1}\partial_{\pm}g.$$

The equation (11) then says that this field  $g(\xi^+, \xi^-)$  satisfies the field equations (5) of the principal chiral model.

## 2.2 Yang-Baxter operator $R$

An important role in this paper will be played by certain  $\mathbf{R}$ -linear operator  $R : \mathcal{G} \rightarrow \mathcal{G}$ . In order to define it, it is useful to choose an appropriate basis of  $\mathcal{G}^{\mathbf{C}}$ . We normalize the extension of the Killing-Cartan form  $(.,.)_{\mathcal{G}}$  on  $\mathcal{G}^{\mathbf{C}}$  in such a way that the square of the length of the longest root is equal to two. We pick an orthonormal Hermitian basis  $H^{\mu}$  in the Cartan subalgebra  $\mathcal{H}^{\mathbf{C}}$  of  $\mathcal{G}^{\mathbf{C}}$  with respect to the Killing Cartan form  $(.,.)_{\mathcal{G}}$ . Consider the root space decomposition of  $\mathcal{G}^{\mathbf{C}}$ :

$$\mathcal{G}^{\mathbf{C}} = \mathcal{H}^{\mathbf{C}} \oplus (\oplus_{\alpha \in \Phi} \mathbf{C}E^{\alpha}),$$

where  $\alpha$  runs over the space  $\Phi$  of all roots  $\alpha \in \mathcal{H}^{*\mathbf{C}}$ . The step generators  $E^\alpha$  fulfil

$$\begin{aligned} [H^\mu, E^\alpha] &= \alpha(H^\mu)E^\alpha, \quad (E^\alpha)^\dagger = E^{-\alpha}; \\ [E^\alpha, E^{-\alpha}] &= \alpha^\vee, \quad [\alpha^\vee, E^{\pm\alpha}] = \pm 2E^{\pm\alpha}, \quad (E^\alpha, E^{-\alpha})_{\mathcal{G}} = \frac{2}{|\alpha|^2}. \end{aligned}$$

The element  $\alpha^\vee \in \mathcal{H}^{\mathbf{C}}$  is called the coroot of the root  $\alpha$ . Thus the (ordinary Cartan-Weyl) basis of the complex Lie algebra  $\mathcal{G}^{\mathbf{C}}$  is  $(H^\mu, E^\alpha)$ ,  $\alpha \in \Phi$ .

A basis of the real Lie algebra  $\mathcal{G}$  can be then chosen as  $(T^\mu, B^\alpha, C^\alpha)$ ,  $\alpha > 0$  where

$$T^\mu = iH^\mu, \quad B^\alpha = \frac{i}{\sqrt{2}}(E^\alpha + E^{-\alpha}), \quad C^\alpha = \frac{1}{\sqrt{2}}(E^\alpha - E^{-\alpha}).$$

Define the  $\mathbf{R}$ -linear operator  $R : \mathcal{G} \rightarrow \mathcal{G}$  as follows (cf. [10])

$$RT^\mu = 0, \quad RB^\alpha = C^\alpha, \quad RC^\alpha = -B^\alpha.$$

It is not difficult to check that the operator  $R$  verifies the identity<sup>1</sup>

$$[RX, RY] = R([RX, Y] + [X, RY]) + [X, Y], \quad X, Y \in \mathcal{G} \quad (12)$$

and the skew-symmetry condition

$$(RX, Y)_{\mathcal{G}} + (X, RY)_{\mathcal{G}} = 0.$$

Moreover, the antisymmetric bracket

$$[X, Y]_R \equiv [RX, Y] + [X, RY], \quad X, Y \in \mathcal{G} \quad (13)$$

verifies the Jacobi identity by virtue of (12) and, hence, it defines a new Lie algebra structure  $\mathcal{G}_R \equiv (\mathcal{G}, [.,.]_R)$  on the vector space  $\mathcal{G}$ .

Denote  $\mathcal{G}^{\mathbf{C}}$  the complexification of  $\mathcal{G}$  and view it as the real Lie algebra. Clearly, the multiplication by the imaginary unit  $i$  is  $\mathbf{R}$ -linear operator from

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<sup>1</sup>In [10], there is a minus sign in front of the last term  $[X, Y]$  in the right hand side of Eq. (12). This circumstance reflects the fact that there are three non-equivalent versions of the Yang-Baxter operator: the one introduced in [10], the one discussed here and yet another one (called triangular) where the the last term  $[X, Y]$  in the right hand side of Eq. (12) is simply absent .

$\mathcal{G}^{\mathbf{C}} \rightarrow \mathcal{G}^{\mathbf{C}}$  and thus  $(R - i)$  can be understood as  $\mathbf{R}$ -linear operator from  $\mathcal{G} \subset \mathcal{G}^{\mathbf{C}} \rightarrow \mathcal{G}^{\mathbf{C}}$ . Using the identity (12), it can be easily verified that the operator  $(R - i) : \mathcal{G} \rightarrow \mathcal{G}^{\mathbf{C}}$  is in fact an injective homomorphism between the real Lie algebras  $\mathcal{G}_R$  and  $\mathcal{G}^{\mathbf{C}}$  and it thus permits to view  $\mathcal{G}_R$  as the real subalgebra of  $\mathcal{G}^{\mathbf{C}}$ .

The subgroup  $G_R$  of  $G^{\mathbf{C}}$ , integrating the Lie subalgebra  $\mathcal{G}_R$  of  $\mathcal{G}^{\mathbf{C}}$ , turns out to be nothing but the so called group  $AN$ . Recall, that an element  $b$  of  $AN$  can be uniquely represented by means of the exponential map as follows

$$b = e^{\phi} \exp[\Sigma_{\alpha > 0} v_{\alpha} E^{\alpha}] \equiv e^{\phi} n.$$

Here  $\alpha$ 's denote the roots of  $\mathcal{G}^{\mathbf{C}}$ ,  $v_{\alpha}$  are complex numbers,  $E^{\alpha}$  are the step generators and  $\phi$  is an Hermitian element of the Cartan subalgebra of  $\mathcal{G}^{\mathbf{C}}$ . In particular, if  $G^{\mathbf{C}} = SL(n, \mathbf{C})$ , the group  $AN$  can be identified with the group of upper triangular matrices of determinant 1 and with positive real numbers on the diagonal.

## 2.3 Poisson-Lie symmetry

Poisson-Lie symmetry of a nonlinear  $\sigma$ -model is the concept which was originally introduced in the context of the so called T-duality [4]. However, as we shall see in this paper, the Poisson-Lie symmetry is an interesting structure also outside of the T-duality story. In what follows, we shall first review this crucial concept in general and then we shall analyse the particular case of  $\sigma$ -models on simple compact group targets.

As it is well-known, a two-dimensional non-linear  $\sigma$ -model is a field theory canonically associated to a metric (symmetric tensor)  $G_{ij}$  and a two-form (antisymmetric tensor)  $B_{ij}$  on some manifold  $M$ . Its action in some local coordinates  $x^i$  is given by

$$S = \int_W (G_{ij}(x) + B_{ij}(x)) \partial_+ x^i \partial_- x^j \equiv \int_W E_{ij}(x) \partial_+ x^i \partial_- x^j, \quad (14)$$

Let  $G$  be a Lie group which infinitesimally acts on the manifold  $M$  by means of the Lie algebra homomorphism  $v : \mathcal{G} \rightarrow Vect(M)$ . In particular, to a basis  $T^a$  of  $\mathcal{G}$  there are associated vector fields  $v^a(x) \equiv v^{ai}(x) \partial_{x^i}$ . We say that the  $\sigma$ -model (14) is Poisson-Lie symmetric with respect to the  $G$ -action on  $M$  if it holds

$$\mathcal{L}_{v^a} E_{ij} = -\tilde{f}_{bc}^a v^{bm} v^{cn} E_{mj} E_{in}. \quad (15)$$

Here  $\mathcal{L}_{v^a}$  means the Lie derivative of the tensor  $E_{ij}$  with respect to  $v^a$  and  $\tilde{f}_{bc}^a$  are the structure constant of some Lie algebra  $\tilde{\mathcal{G}}$  with the dimension equal to that of  $\mathcal{G}$ . If the structure constants  $\tilde{f}_{bc}^a$  vanish, then we say that the  $\sigma$ -model is symmetric in the ordinary sense.

Let us motivate the defining relation (15) of the Poisson-Lie symmetry. For that, consider the variation of the action (14) with respect to the  $G$ -transformations with the *world-sheet dependent* parameters  $\kappa_a(\sigma, \tau)$ :

$$\delta S \equiv S(x + \kappa_a v^a) - S(x) = \int_W \kappa_a \mathcal{L}_{v^a} E_{ij} \partial_+ x^i \partial_- x^j + \int_W J^a \wedge d\kappa_a.$$

Here the world-sheet current one-forms  $J_a$  are

$$J^a(x) = -v^{ai}(x) E_{ij}(x) \partial_- x^j d\xi^- + v^{ai}(x) E_{ji}(x) \partial_+ x^j d\xi^+. \quad (16)$$

With the help of (15) and the integration per partes, the variation  $\delta S$  can be rewritten as

$$\delta S = \int_W \kappa_a (dJ^a - \frac{1}{2} \tilde{f}_{bc}^a J^b \wedge J^c).$$

For every solution of the field equations the variation  $\delta S$  vanishes, which means that the 1-forms  $J^a$  are components of a non-Abelian  $\tilde{\mathcal{G}}$ -valued flat connection verifying the following zero curvature condition

$$dJ^a - \frac{1}{2} \tilde{f}_{bc}^a J^b \wedge J^c = 0. \quad (17)$$

In particular, when the group  $G$  acts on  $M$  transitively, then the zero curvature condition (17) coincides with the complete set of the field equations of the Poisson-Lie symmetric  $\sigma$ -model (14). This remark will be relevant in what follows.

Now consider the case where the  $\sigma$ -model target  $M$  is a simple compact group  $G$ , the action of  $G$  on itself is the standard right multiplication and the Lie algebra  $\tilde{\mathcal{G}}$  is just the Lie algebra  $\mathcal{G}_R$  defined in the previous subsection with the help of the Yang-Baxter operator  $R$ . The most general form of the Poisson-Lie symmetric  $\sigma$ -model in this case was found in [4, 5] and it reads

$$S_\varepsilon(E)(g) = \int_W (g^{-1} \partial_+ g, (E_g - \varepsilon R)^{-1} g^{-1} \partial_- g)_{\mathcal{G}}. \quad (18)$$

Here  $\varepsilon$  is the Poisson-Lie deformation parameter,  $E_g \equiv Ad_{g^{-1}} E Ad_g$  and  $E$  is any  $\mathbf{R}$ -linear operator  $E : \mathcal{G} \rightarrow \mathcal{G}$ . Note that, in the language which uses the



operator  $R$ , the  $\sigma$ -model action can be written in a basis independent way. The same is true for the  $\mathcal{G}_R$ -valued world-sheet current 1-form  $J(g)$ . Indeed, the general formula (16) can be now written as

$$\begin{aligned} J(g) &= J_+(g)d\xi^+ + J_-(g)d\xi^- = \\ &= -(E_g^t + \varepsilon R)^{-1}g^{-1}\partial_+g \, d\xi^+ + (E_g - \varepsilon R)^{-1}g^{-1}\partial_-g \, d\xi^-, \end{aligned} \quad (19)$$

where  $E^t : \mathcal{G} \rightarrow \mathcal{G}$  is the transposition of  $E$  with respect to the Killing-Cartan form:

$$(EX, Y)_{\mathcal{G}} = (X, E^t Y)_{\mathcal{G}}, \quad X, Y \in \mathcal{G}.$$

The field equations of the model (18) has the form of the zero curvature condition in the algebra  $\tilde{\mathcal{G}} = \mathcal{G}_R$  (cf. Eq. (17)):

$$\partial_+ J_-(g) - \partial_- J_+(g) + \varepsilon[J_-(g), J_+(g)]_R = 0. \quad (20)$$

The meaning of the Poisson-Lie deformation parameter  $\varepsilon$  is now clear. It simply rescales the commutator of the algebra  $\mathcal{G}_R$ , or, in other words, it replaces the structure constants  $\tilde{f}_{bc}^a$  by the structure constants  $\varepsilon \tilde{f}_{bc}^a$ .

So far we have established that to every solution  $g$  of the Poisson-Lie symmetric model (18) there is a  $\mathcal{G}_R$ -valued flat connection  $J(g)$ . Now we are going to show that there is also a natural  $\mathcal{G}^{\mathbf{C}}$ -valued flat connection  $B^\varepsilon(g)$  associated to the same  $g$ . It is easy to construct  $B^\varepsilon(g)$ ; its chiral components read

$$B_\pm^\varepsilon(g) = \partial_\pm g g^{-1} + g \varepsilon (R - i) J_\pm(g) g^{-1}.$$

In words: we have first injected the flat connection  $J(g)$  into  $\mathcal{G}^{\mathbf{C}}$  with the help of the homomorphism  $\varepsilon(R - i)$  (cf. Sec. 2.2) and then we have performed a  $g$ -gauge transformation. Both operations evidently preserve the flatness. The explicit formulae for  $B_\pm^\varepsilon(g)$  read

$$B_+^\varepsilon(g) = (E^t + i\varepsilon)(E^t + \varepsilon R_{g^{-1}})^{-1} \partial_+ g g^{-1}, \quad B_-^\varepsilon(g) = (E - i\varepsilon)(E - \varepsilon R_{g^{-1}})^{-1} \partial_- g g^{-1}. \quad (21)$$

Here  $R_{g^{-1}} \equiv Ad_g R Ad_{g^{-1}}$ . Thus, if  $g$  is a solution of the model (18) then  $B_\pm^\varepsilon(g)$  fulfil the following zero curvature condition

$$\partial_+ B_-^\varepsilon(g) - \partial_- B_+^\varepsilon(g) + [B_-^\varepsilon(g), B_+^\varepsilon(g)] = 0,$$

where the commutator is that of the Lie algebra  $\mathcal{G}^{\mathbf{C}}$ .

### 3 Lax pairs from the Poisson-Lie symmetry

#### 3.1 The Yang-Baxter $\sigma$ -model

We have seen in the previous subsection that there are as many Poisson-Lie symmetric  $\sigma$ -models on  $G$  as are the operators  $E : \mathcal{G} \rightarrow \mathcal{G}$ . Among them, there is a distinguished choice  $E = I$  where the action (18) becomes the action (2). In [6], we have called this choice the Yang-Baxter  $\sigma$ -model. The reason why it deserves a special name is the fact that the Yang-Baxter model is not only Poisson-Lie symmetric with respect to the right action of  $G$  on itself but is also symmetric in the standard way with respect to the left action of  $G$  on itself. Note that the principal chiral model (6) is bisymmetric, that is, it is symmetric in the standard way with respect to the both left and right actions of  $G$  on itself. The Yang-Baxter  $\sigma$ -model can be then interpreted as the right Poisson-Lie deformation of the principal chiral model which leaves the ordinary left symmetry intact.

The crucial observation, which has triggered our interest in the problem of proving the integrability of the Yang-Baxter  $\sigma$ -model, is the similarity of the form of the Lax connection (7),(8) associated to the principal chiral model and the flat connection (21) associated to the Yang-Baxter model. Indeed, for the Yang-Baxter choice  $E = I$ , Eq.(21) gives

$$B_{\pm}^{\varepsilon}(g_{\varepsilon}) = \frac{1}{1 \mp i\varepsilon} \frac{1 + \varepsilon^2}{I \pm \varepsilon R_{g_{\varepsilon}^{-1}}} \partial_{\pm} g_{\varepsilon} g_{\varepsilon}^{-1} \equiv -\frac{1}{1 \mp i\varepsilon} u_{\pm}^{\varepsilon}. \quad (22)$$

If, for each  $\varepsilon$ , we succeed to find a solution  $g_{\varepsilon}$  of the Yang-Baxter model (2) in such a way that the quantities  $u_{\pm}^{\varepsilon} \in \mathcal{G}$  do not depend on  $\varepsilon$ , then Eq.(22) becomes Eq.(10), or, in other words, becomes the Zakharov-Mikhailov Lax pair (10) of the principal chiral model for  $\lambda = -i\varepsilon$ . Following the reasoning in Section 2.1, we obtain in this way a solution  $g(\xi^+, \xi^-)$  of the principal chiral model such that

$$u_{\pm} = g^{-1} \partial_{\pm} g.$$

In fact, we shall see in Section 3.3 that every solution  $g$  of the principal chiral model can be obtained in this way. Let us not anticipate things too much, however, and let us concentrate on the most important aspect of the story at the moment. Namely, we have observed that there is a relation between the flat  $\mathcal{G}^{\mathbb{C}}$ -valued connection (22) canonically associated to the Yang-Baxter

model and the Lax connection (10) of the principal chiral model. We shall see that a relation of this type will hold also in a more general context and it will permit us to find the Lax pair of the Yang-Baxter  $\sigma$ -model itself.

### 3.2 A bi-Yang-Baxter $\sigma$ -model

In Section 3.1, we have observed that there exists a *Poisson-Lie deformed* (i.e. the Yang-Baxter)  $\sigma$ -model such that the deformation parameter  $\varepsilon$  can be interpreted as the spectral parameter occurring in the Lax pair of the *non-deformed* (i.e. the principal chiral) model. This observation motivates us to look for a Lax pair of the Yang-Baxter  $\sigma$ -model as follows: we shall look for a further Poisson-Lie deformation of the Yang-Baxter  $\sigma$ -model itself (i.e. a two-parameter deformation of the principal chiral model) such that the new deformation parameter  $\eta$  would be related to the spectral parameter of the Lax pair of the Yang-Baxter  $\sigma$ -model. Note that in this way we change the role of the Yang-Baxter  $\sigma$ -model in the whole story. Indeed, in the previous section it played the role of the *deformed* model and in this one it will play the role of the *non-deformed* model. This may seem paradoxical but remember that now we are going to have two distinct Poisson-Lie deformations in game.

How to deform in the Poisson-Lie way the Yang-Baxter model which is already the Poisson-Lie deformation of the principal chiral model? Well, we have to realize that the Yang-Baxter model is the result of the Poisson-Lie deformation of the *right*  $G$ -symmetry of the principal chiral model. Moreover, this right deformation leaves the ordinary *left* symmetry intact. Thus we can deform in a Poisson-Lie way the left symmetry of the Yang-Baxter model and see whether the structure of such a left-deformed model will permit us to construct the Yang-Baxter Lax pair (3). We shall see that this approach indeed works.

We now look for the left deformation of the Yang-Baxter  $\sigma$ -model which leaves the right Poisson-Lie symmetry intact, or, in other words, we look for a  $\sigma$ -model on the simple compact group target  $G$  which is Poisson-Lie symmetric from both left and right. We know already that all right Poisson-Lie symmetric models must have the form (18), that is

$$S_{\varepsilon_r}(E)(g) = \int_W (g^{-1} \partial_+ g, (E_g - \varepsilon_r R)^{-1} g^{-1} \partial_- g)_{\mathcal{G}}. \quad (23)$$

Here the deformation parameter is denoted as  $\varepsilon_r$  in order to stress that it corresponds to the *right* Poisson-Lie symmetry. Our problem is simply to find the operator  $E : \mathcal{G} \rightarrow \mathcal{G}$  in such a way that the right Poisson-Lie symmetric model (23) would be simultaneously left Poisson-Lie symmetric. Note that the diffeomorphism  $g \rightarrow g^{-1}$  interchanges the left action of the group  $G$  on itself with the right action. This means that the right Poisson-Lie symmetric model (23) will be also the left Poisson-Lie symmetric if it exists a couple of operators  $E, E' : \mathcal{G} \rightarrow \mathcal{G}$  such that

$$S_{\varepsilon_r}(E)(g^{-1}) = \int_W (g^{-1} \partial_+ g, (E'_g - \varepsilon_l R)^{-1} g^{-1} \partial_- g)_g \equiv S_{\varepsilon_l}(E')(g). \quad (24)$$

Here  $\varepsilon_r$  and  $\varepsilon_l$  are the deformation parameters corresponding to the right and the left Poisson-Lie symmetry, respectively. It is easy to find  $E$  and  $E'$  which solve Eq. (24):

$$E = I - \varepsilon_l R, \quad E' = I - \varepsilon_r R.$$

Thus the  $\sigma$ -model enjoying simultaneously the left and the right Poisson-Lie symmetry is defined by the following action

$$S_{\varepsilon_r, \varepsilon_l}(g) \equiv S_{\varepsilon_r}(I - \varepsilon_l R)(g) = \int_W (g^{-1} \partial_+ g, (I - \varepsilon_l R_g - \varepsilon_r R)^{-1} g^{-1} \partial_- g)_g. \quad (25)$$

We shall call it a bi-Yang-Baxter model. Note that it is the  $\varepsilon_l$ -deformation of the Yang-Baxter model (2) and also the two-parametric deformation of the principal chiral model (6).

The bi-Yang-Baxter  $\sigma$ -model is, in particular, right Poisson-Lie symmetric. Therefore we may use the explicit formula (21) for the  $\mathcal{G}^{\mathbf{C}}$ -valued flat connection  $B^{\varepsilon_r}(E)$  for  $E = I - \varepsilon_l R$ . Thus, if  $g_{\varepsilon_r, \varepsilon_l}$  is a solution of the field equations of the bi-Yang-Baxter  $\sigma$ -model (25), then the following  $\mathcal{G}^{\mathbf{C}}$ -valued fields

$$B_{\pm}(g_{\varepsilon_r, \varepsilon_l}) = (I \pm \varepsilon_l R \pm i \varepsilon_r)(I \pm \varepsilon_l R \pm \varepsilon_r R g^{-1})^{-1} \partial_{\pm} g_{\varepsilon_r, \varepsilon_l} g_{\varepsilon_r, \varepsilon_l}^{-1} \equiv (I \pm \varepsilon_l R \pm i \varepsilon_r) f v_{\pm} \quad (26)$$

are the chiral components of the flat connection  $B^{\varepsilon_r}(I - \varepsilon_l R)$ . Here  $f$  is a normalization parameter that we have introduced for later convenience.

Taking motivation from Eqs. (26) and (22), we shall look for the Lax pair of the Yang-Baxter  $\sigma$ -model (2) in the following form

$$A^{\varepsilon}(-i\eta)_{\pm} = \left( I \pm \varepsilon_l(\varepsilon, \eta) R \pm i \varepsilon_r(\varepsilon, \eta) \right) f(\varepsilon, \eta) v_{\pm}^{\varepsilon}. \quad (27)$$

Here  $\varepsilon$  is the deformation parameter occurring in the Yang-Baxter action (2),  $-\eta$  is the imaginary part of the spectral parameter  $\lambda$ ,  $\varepsilon_{l,r}(\varepsilon, \eta)$  and  $f(\varepsilon, \eta)$  are functions to be determined and  $v_{\pm}^{\varepsilon}$  can depend only on  $\varepsilon$  but not on  $\eta$ .

The  $\mathcal{G}^{\mathbf{C}}$ -valued zero curvature condition for the Lax pair  $A^{\varepsilon}(-i\eta)_{\pm}$  must read

$$\partial_+ A_-^{\varepsilon}(-i\eta) - \partial_- A_+^{\varepsilon}(-i\eta) + [A_-^{\varepsilon}(-i\eta), A_+^{\varepsilon}(-i\eta)] = 0. \quad (28)$$

Using the fact that the quantities  $v_{\pm}$  are  $\mathcal{G}$ -valued, the  $\mathcal{G}^{\mathbf{C}}$ -valued condition (28) can be rewritten in terms of two  $\mathcal{G}$ -valued conditions (i.e. the real and the imaginary parts of Eq. (28)):

$$\partial_+ v_-^{\varepsilon} + \partial_- v_+^{\varepsilon} + \varepsilon_l f[v_-^{\varepsilon}, v_+^{\varepsilon}]_R = 0, \quad (29)$$

$$(I - \varepsilon_l R) \partial_+ v_-^{\varepsilon} - (I + \varepsilon_l R) \partial_- v_+^{\varepsilon} + f[(I - \varepsilon_l R) v_-^{\varepsilon}, (I + \varepsilon_l R) v_+^{\varepsilon}] + \varepsilon_r^2 f[v_-^{\varepsilon}, v_+^{\varepsilon}] = 0. \quad (30)$$

Acting with the operator  $R$  on the first of these two conditions, using the identity (12) and inserting the result in the second condition, we can rewrite Eq. (30): as

$$\partial_+ v_-^{\varepsilon} - \partial_- v_+^{\varepsilon} + (\varepsilon_r^2 - \varepsilon_l^2 + 1) f[v_-^{\varepsilon}, v_+^{\varepsilon}] - \varepsilon_l f[R v_-^{\varepsilon}, v_+^{\varepsilon}] + \varepsilon_l f[v_-^{\varepsilon}, R v_+^{\varepsilon}] = 0. \quad (31)$$

Because  $v_{\pm}^{\varepsilon}$  do not depend on  $\eta$ , Eqs. (29) and (31) imply that expressions  $\varepsilon_l f$  and  $(\varepsilon_r^2 - \varepsilon_l^2 + 1) f$  do not depend on  $\eta$  either. As far as the dependence on  $\varepsilon$ , we can determine it by considering the equations of motions and the Bianchi identities of the Yang-Baxter model (2). The equations of motion have been already given in (20)

$$\partial_+ J_- - \partial_- J_+ + \varepsilon [J_-, J_+]_R = 0, \quad (32)$$

where, following Eq.(19),

$$J_{\pm} = \mp (I \pm \varepsilon R)^{-1} g^{-1} \partial_{\pm} g. \quad (33)$$

Let us rewrite Eq.(33) equivalently as

$$g^{-1} \partial_{\pm} g = \mp (I \pm \varepsilon R) J_{\pm}.$$

Evidently, it holds

$$\partial_+ (g^{-1} \partial_- g) - \partial_- (g^{-1} \partial_+ g) - [g^{-1} \partial_- g, g^{-1} \partial_+ g] = 0,$$

hence also

$$(I - \varepsilon R)\partial_+ J_- + (I + \varepsilon R)\partial_- J_+ + [(I - \varepsilon R)J_-, (I + \varepsilon R)J_+] = 0. \quad (34)$$

The equation (34) is the Bianchi identity. Acting with the operator  $R$  on the equations of motion (32), using the identity (12) and inserting the result into (34), we can rewrite the Bianchi identity as

$$\partial_+ J_- + \partial_- J_+ + (1 - \varepsilon^2)[J_-, J_+] - \varepsilon[RJ_-, J_+] + \varepsilon[J_-, RJ_+] = 0. \quad (35)$$

Now the comparison of Eq.(29) with Eq.(32) and of Eq.(31) with Eq.(35) leads to the following identifications

$$v_\pm^\varepsilon = \mp J_\pm = (I \pm \varepsilon R)^{-1} g^{-1} \partial_\pm g,$$

$$\varepsilon_l f = -\varepsilon; \quad (36)$$

$$(\varepsilon_r^2 - \varepsilon_l^2 + 1)f = \varepsilon^2 - 1. \quad (37)$$

The equation (36) means that the Lax pair ansatz (27) can be rewritten as

$$A^\varepsilon(-i\eta)_\pm = \left( f(\varepsilon, \eta) \mp \varepsilon R \pm i f(\varepsilon, \eta) \varepsilon_r(\varepsilon, \eta) \right) v_\pm^\varepsilon. \quad (38)$$

Thus we see that the coefficient in front of  $R$  does not depend on the spectral parameter  $\eta$  which makes possible to refine the ansatz (27) as

$$A^\varepsilon(\lambda)_\pm = \left( F(\varepsilon) \mp \varepsilon R + \frac{G(\varepsilon)}{1 \pm \lambda} \right) v_\pm^\varepsilon,$$

where  $\lambda$  is the full-fledged complex spectral parameter. For  $\lambda = -i\eta$ , this gives

$$A^\varepsilon(-i\eta)_\pm = \left( F(\varepsilon) + \frac{G(\varepsilon)}{1 + \eta^2} \mp \varepsilon R \pm i \frac{G(\varepsilon)\eta}{1 + \eta^2} \right) v_\pm^\varepsilon.$$

Comparing with (38), we obtain

$$f(\varepsilon, \eta) = F(\varepsilon) + \frac{G(\varepsilon)}{1 + \eta^2}, \quad f(\varepsilon, \eta) \varepsilon_r(\varepsilon, \eta) = \frac{G(\varepsilon)\eta}{1 + \eta^2}. \quad (39)$$

Inserting the conditions (39) into Eq.(37), we infer

$$\frac{G^2 + G(2F + 1 - \varepsilon^2)}{\eta^2 + 1} = \varepsilon^2 + F(\varepsilon^2 - 1) - F^2 = 0. \quad (40)$$

Clearly, the equality in Eq.(40) can take place only if it holds simultaneously

$$G + 2F = \varepsilon^2 - 1, \quad \varepsilon^2 + F(\varepsilon^2 - 1) - F^2 = 0.$$

There are two solutions of the quadratic equation for  $F$ , however, only one of them has the correct limit  $\varepsilon \rightarrow 0$  when the Yang-Baxter Lax pair should become the Zakharov-Mikhailov principal chiral Lax pair (4). It reads

$$F(\varepsilon) = \varepsilon^2, \quad G(\varepsilon) = -(1 + \varepsilon^2).$$

Finally, the Lax pair of the Yang-Baxter  $\sigma$ -model (2) reads

$$A_{\pm}^{\varepsilon}(\lambda) \equiv \mp \left( \varepsilon^2 \mp \varepsilon R - \frac{1 + \varepsilon^2}{1 \pm \lambda} \right) J_{\pm} \quad (41)$$

or

$$A_{\pm}^{\varepsilon}(\lambda) = \left( \varepsilon^2 \mp \varepsilon R - \frac{1 + \varepsilon^2}{1 \pm \lambda} \right) (I \pm \varepsilon R)^{-1} g^{-1} \partial_{\pm} g. \quad (42)$$

As in the case of the principal chiral model, there are two ways of interpretation of the Lax pair. In the first formulation, we consider two quantities  $J_{\pm} : W \rightarrow \mathcal{G}$  such that the quantities  $A_{\pm}^{\varepsilon}(\lambda) : W \rightarrow \mathcal{G}^{\mathbf{C}}$  given by (41) verify for each complex  $\lambda$  the zero curvature condition

$$\partial_+ A_-^{\varepsilon}(\lambda) - \partial_- A_+^{\varepsilon}(\lambda) + [A_-^{\varepsilon}(\lambda), A_+^{\varepsilon}(\lambda)] = 0. \quad (43)$$

Then we conclude that  $J_{\pm}$  must be of the form (33), where  $g : W \rightarrow G$  satisfies the equations of motion of the Yang-Baxter  $\sigma$ -model (2). In the second formulation, we consider a solution  $g : W \rightarrow G$ , which satisfies the equations of motion of (2), and we associate to it the quantities  $A_{\pm}^{\varepsilon}(\lambda) : W \rightarrow \mathcal{G}^{\mathbf{C}}$  according to Eq.(42). Then we conclude that those quantities verify for each complex  $\lambda$  the zero curvature condition (43).

### 3.3 Extended solutions

The concept of the extended solution [11] plays an important role in the studies of the principal chiral model, in particular in connection with the so called dressing symmetries [12, 10, 3, 2, 7, 9]. In this paper, we find a new application of this concept by showing that the extended solutions  $l_0(\lambda)$  of the principal chiral model encapsulate also the dynamics and the symmetry structure of the Yang-Baxter  $\sigma$ -model (2).

In order to define the *extended* solution let us make more precise the notion of the *ordinary* solution of the principal chiral model. For that, we first remark that if  $a \in G$  and  $g(\xi^+, \xi^-)$  is a solution of the equation of motion (5) then  $ag(\xi^+, \xi^-)$  is also the solution of the same equation of motion. In what follows, we shall be interested in the classes of equivalences of the solutions of Eq.(5); that is, two solutions  $g_1(\xi^+, \xi^-)$  and  $g_2(\xi^+, \xi^-)$  will be considered equivalent if there is  $a \in G$  such that  $g_1(\xi^+, \xi^-) = ag_2(\xi^+, \xi^-)$ . Any such class of equivalence will be called the ordinary solution of the principal chiral model and will be canonically represented by the solution  $g(\xi^+, \xi^-)$  of Eq. (5) fulfilling  $g(0, 0) = e$ . Here  $e$  denote the unit element of  $G$  and  $(0, 0)$  is the point of the worldsheet  $W$  for which  $\xi^+ = \xi^- = 0$ .

Let  $g_0 : W \rightarrow G$  be an ordinary solution of the principal chiral model and consider the associated Zakharov-Mikhailov Lax pair (4). Clearly, the zero curvature condition (9) on a simply connected world-sheet  $W$  implies that there is a unique map  $l_0(\lambda)$  from  $W$  to the complexified group  $G^{\mathbb{C}}$  such that

$$-l_0^{-1}(\lambda)\partial_{\pm}l_0(\lambda) = A_{\pm}^0(\lambda), \quad (44)$$

$$l_0(\lambda)(0, 0) = e. \quad (45)$$

Here Eq. (45) is an initial condition which ensures the unicity of  $l_0$ . The map  $l_0(\lambda) : W \rightarrow G^{\mathbb{C}}$  is often called the extended solution of the principal chiral model [11, 2, 7], because for a particular value  $\lambda = 0$  it reduces just to the ordinary solution  $g_0$ , i.e.

$$l_0(0)(\xi^+, \xi^-) = g_0(\xi^+, \xi^-).$$

Similarly as in the principal chiral case, if  $a \in G$  and  $g_{\varepsilon}(\xi^+, \xi^-)$  is a solution of the equations of motion (32) and (33) of the Yang-Baxter model then  $ag_{\varepsilon}(\xi^+, \xi^-)$  is also the solution of the same equations. As before, we shall be interested in the classes of equivalences of the solutions of Eqs. (32) and (33) defined as the orbits of the left  $G$ -action just described. Any such class of equivalence will be called the ordinary solution of the Yang-Baxter model and will be canonically represented by the solution  $g_{\varepsilon}(\xi^+, \xi^-)$  of Eqs. (32) and (33) fulfilling  $g_{\varepsilon}(0, 0) = e$ .

As we already know, if  $g_{\varepsilon}(\xi^+, \xi^-)$  solves the equation of motions (32) and (33) of the Yang-Baxter model, than the corresponding Lax pair (3) solves



the zero-curvature condition and gives rise to the unique map  $l_\varepsilon(\lambda)$  from  $W$  to  $G^{\mathbf{C}}$  such that

$$-l_\varepsilon^{-1}(\lambda)\partial_\pm l_\varepsilon(\lambda) = A_\pm^\varepsilon(\lambda), \quad (46)$$

$$l_\varepsilon(\lambda)(0, 0) = e. \quad (47)$$

We shall call the map  $l_\varepsilon(\lambda) : W \rightarrow G^{\mathbf{C}}$  the extended solution of the Yang-Baxter model. As in the principal chiral case, it holds evidently

$$A_\pm^\varepsilon(0) = -g_\varepsilon^{-1}\partial_\pm g_\varepsilon$$

hence again, for  $\lambda = 0$ , the extended solution reduces to the ordinary one:

$$l_\varepsilon(0)(\xi^+, \xi^-) = g_\varepsilon(\xi^+, \xi^-).$$

Remarkably, we can extract from the extended solution  $l_\varepsilon(\lambda)$  of the Yang-Baxter  $\sigma$ -model also certain ordinary solution of the principal chiral model. Indeed, we have:

**Theorem 1:** Consider an ordinary solution  $g_\varepsilon : W \rightarrow G$  of the Yang-Baxter model (2) and its associated *Yang-Baxter* extended solution  $l_\varepsilon(\lambda) : W \rightarrow G^{\mathbf{C}}$ . Then the product  $l_\varepsilon(\lambda^{-1})l_\varepsilon^{-1}(0)$  turns out to be the *principal chiral* extended solution  $l_0(\lambda)$  associated to certain ordinary solution  $g_0$  of the principal chiral model.

**Proof:** From the defining relation

$$l_0(\lambda) \equiv l_\varepsilon(\lambda^{-1})l_\varepsilon^{-1}(0), \quad (48)$$

we first obtain

$$l_0(\lambda)^{-1}\partial_\pm l_0(\lambda) = l_\varepsilon(0)\left(l_\varepsilon(\lambda^{-1})^{-1}\partial_\pm l_\varepsilon(\lambda^{-1}) - l_\varepsilon(0)^{-1}\partial_\pm l_\varepsilon(0)\right)l_\varepsilon^{-1}(0).$$

Now we use (3) and (46) to write

$$l_\varepsilon(\lambda^{-1})^{-1}\partial_\pm l_\varepsilon(\lambda^{-1}) - l_\varepsilon(0)^{-1}\partial_\pm l_\varepsilon(0) = -A_\pm^\varepsilon(\lambda^{-1}) + A_\pm^\varepsilon(0) = \pm \frac{1 + \varepsilon^2}{1 \pm \lambda} J_\pm.$$

Thus we obtain

$$l_0(\lambda)^{-1}\partial_\pm l_0(\lambda) = \pm \frac{1 + \varepsilon^2}{1 \pm \lambda} l_\varepsilon(0) J_\pm l_\varepsilon^{-1}(0). \quad (49)$$

From the relation (49), considered at  $\lambda = 0$ , we obtain

$$l_0(0)^{-1}\partial_{\pm}l_0(0) = \pm(1 + \varepsilon^2)l_{\varepsilon}(0)J_{\pm}l_{\varepsilon}^{-1}(0). \quad (50)$$

Because the map  $l_{\varepsilon}(0) = g_{\varepsilon}$  takes values in the subgroup  $G$  of  $G^{\mathbf{C}}$ , the expressions  $\pm(1 + \varepsilon^2)l_{\varepsilon}(0)J_{\pm}l_{\varepsilon}^{-1}(0)$  take values in the subalgebra  $\mathcal{G}$  of  $\mathcal{G}^{\mathbf{C}}$ . The condition (47) implies also that  $l_0(0)(0,0) = e$ , thus we see from (50) that the map  $l_0(0)$  takes values in  $G$ . We set

$$g_0 \equiv l_0(0) = l_{\varepsilon}(\infty)l_{\varepsilon}^{-1}(0).$$

With the help of Eq.(50), we can rewrite Eq.(49) as

$$l_0(\lambda)^{-1}\partial_{\pm}l_0(\lambda) = \frac{g_0^{-1}\partial_{\pm}g_0}{1 \pm \lambda}. \quad (51)$$

Finally, from Eqs.(8),(44) and (57), we conclude that  $g_0(\xi^+, \xi^-)$  is the solution of the principal chiral field equations (5). Obviously,  $l_0(\lambda)$  is the extended solution associated to it.

#

Theorem 1 gives the construction of a map  $\Xi_{\varepsilon}$  from the space of ordinary solutions of the Yang-Baxter model into the space of ordinary solutions of the principal chiral model. Indeed, the map  $\Xi_{\varepsilon}$  is given by

$$\Xi_{\varepsilon}(g_{\varepsilon}) \equiv l_{\varepsilon}(\infty)l_{\varepsilon}^{-1}(0),$$

where  $l_{\varepsilon}$  is the extended solution associated to an ordinary solution  $g_{\varepsilon}$ . We wish to prove that the map  $\Xi_{\varepsilon}$  is in fact bijective and we wish also to write explicitly the inverse map  $\Xi_{\varepsilon}^{rec}$ . In order to do that we need the theorem of Iwasawa [13]. This well-known group-theoretical theorem affirms the existence and the uniqueness of a map  $Iw : G^{\mathbf{C}} \rightarrow G$  such that, for every  $l \in G^{\mathbf{C}}$ , the product  $lIw(l)$  belong to  $AN$ . (Recall that the real subgroup  $AN$  of the group  $G^{\mathbf{C}}$  was introduced at the end of Section 2.2 as the group integrating the Lie algebra  $\mathcal{G}_R$ .)

**Theorem 2:** Let  $g_{\varepsilon}$  be an ordinary solution of the Yang-Baxter model and  $l_{\varepsilon}(\lambda)$  the associated extended solution. Denote by  $g_0$  the corresponding ordinary solution  $g_0 = l_{\varepsilon}(\infty)l_{\varepsilon}^{-1}(0)$  of the principal chiral model and  $l_0(\lambda)$  the associated extended solution. Then it holds

$$g_{\varepsilon} = Iw(l_0(-i\varepsilon)), .$$

**Proof:** Consider the extended solution  $l_\varepsilon(\lambda)$  and pick  $\lambda = i\varepsilon^{-1}$ . From (41) and (46), we infer

$$l_\varepsilon^{-1}(i\varepsilon^{-1})\partial_\pm l_\varepsilon(i\varepsilon^{-1}) = \pm\left(\varepsilon^2 \mp \varepsilon R - \frac{1 + \varepsilon^2}{1 \pm i\varepsilon^{-1}}\right)J_\pm = -\varepsilon(R - i)J_\pm. \quad (52)$$

Recall that, for any  $v \in \mathcal{G}$ , it holds  $(R - i)v \in \text{Lie}(AN)$  (cf. the discussion in Section 2.2). This fact together with the condition (47) imply that  $l_\varepsilon(i\varepsilon^{-1}) \in AN$ . At the same time we know from (48) that

$$l_0(-i\varepsilon)l_\varepsilon(0) = l_\varepsilon(i\varepsilon^{-1}).$$

Because  $g_\varepsilon = l_\varepsilon(0) \in G$ , the uniqueness of the Iwasawa map  $Iw$  implies that

$$g_\varepsilon = l_\varepsilon(0) = Iw(l_0(-i\varepsilon)).$$

#

The Theorem 2 states, in other words, that

$$\Xi_\varepsilon^{rec}(g_0) = Iw(l_0(-i\varepsilon)),$$

which means that the map  $\Xi_\varepsilon$  is injective. On the other hand, the surjectivity of  $\Xi_\varepsilon$  is the consequence of the following theorem.

**Theorem 3:** Let  $g_0$  be an ordinary solution of the principal chiral model and denote by  $l_0(\lambda)$  the extended solution associated to it. Then the map  $g_\varepsilon : W \rightarrow G$  defined by

$$g_\varepsilon = Iw(l_0(-i\varepsilon)), \quad (53)$$

solves the field equations of the Yang-Baxter model and, moreover, it holds

$$\Xi_\varepsilon(g_\varepsilon) = g_0.$$

**Proof:** Applying the Iwasawa decomposition at every point of the world-sheet, we may write the extended solution  $l_0(-i\varepsilon)$  as

$$l_0(-i\varepsilon) = b_\varepsilon g_\varepsilon^{-1}, \quad (54)$$

where  $g_\varepsilon : W \rightarrow G$  and  $b_\varepsilon : W \rightarrow AN$ . By definition of the Iwasawa map, it holds  $g_\varepsilon = Iw(l_0(-i\varepsilon))$ . We wish to show that  $g_\varepsilon$  solves the field equations (32) and (33) of the Yang-Baxter model. For this, we first use the fact that

$l_0(\lambda)$  is the extended solution of the principal chiral model associated to the ordinary solution  $g_0$ . This implies that

$$\frac{g_0^{-1}\partial_{\pm}g_0}{1 \mp i\varepsilon} = l_0^{-1}(-i\varepsilon)\partial_{\pm}l_0(-i\varepsilon).$$

From this and from Eq.(54) we infer that

$$g_0^{-1}\partial_{\pm}g_0 = (1 \mp i\varepsilon)(g_{\varepsilon}b_{\varepsilon}^{-1}\partial_{\pm}b_{\varepsilon}g_{\varepsilon}^{-1} - \partial_{\pm}g_{\varepsilon}g_{\varepsilon}^{-1}). \quad (55)$$

The expressions  $b_{\varepsilon}^{-1}\partial_{\pm}b_{\varepsilon}$  are clearly in  $\mathcal{G}_R \subset \mathcal{G}^{\mathbf{C}}$  which means that there must exist maps  $J_{\pm} : W \rightarrow \mathcal{G}$  such that

$$b_{\varepsilon}^{-1}\partial_{\pm}b_{\varepsilon} = -\varepsilon(R - i)J_{\pm}. \quad (56)$$

This fact permits to rewrite Eq.(55) as

$$g_0^{-1}\partial_{\pm}g_0 = -(1 \mp i\varepsilon)(\varepsilon g_{\varepsilon}(R - i)J_{\pm}g_{\varepsilon}^{-1} + \partial_{\pm}g_{\varepsilon}g_{\varepsilon}^{-1}). \quad (57)$$

Note that the left hand side of Eq.(57) is  $\mathcal{G}$ -valued but the right hand side is  $\mathcal{G}^{\mathbf{C}}$ -valued. This is possible only if the  $i\mathcal{G}$ -part of the right hand side vanishes:

$$i\varepsilon g_{\varepsilon}(J_{\pm} \pm \varepsilon R J_{\pm} \pm g_{\varepsilon}^{-1}\partial_{\pm}g_{\varepsilon})g_{\varepsilon}^{-1} = 0$$

or

$$J_{\pm} = \mp(1 \pm \varepsilon R)^{-1}g_{\varepsilon}^{-1}\partial_{\pm}g_{\varepsilon}. \quad (58)$$

Because the operator  $(R - i) : \mathcal{G} \rightarrow \mathcal{G}^{\mathbf{C}}$  is the injective homomorphism between the real Lie algebras  $\mathcal{G}_R$  and  $\mathcal{G}^{\mathbf{C}}$ , we infer from Eq.(56) that the quantities  $J_{\pm}$  fulfil the  $\mathcal{G}_R$ -valued zero-curvature condition:

$$\partial_+ J_- - \partial_- J_+ + \varepsilon[J_-, J_+]_R = 0. \quad (59)$$

Comparing Eqs.(59) and (58) with Eqs.(32) and (33) permit to conclude that  $g_{\varepsilon}$  given by Eq.(53) solves the field equations of the Yang-Baxter model.

It remains to show that  $\Xi_{\varepsilon}(g_{\varepsilon}) = g_0$ . Let  $l_{\varepsilon}(\lambda)$  be the extended solution associated to  $g_{\varepsilon}$ , hence  $l_{\varepsilon}(0) = g_{\varepsilon}$ . Moreover, we see from Eqs.(52) and (56) that  $l_{\varepsilon}(i\varepsilon^{-1}) = b_{\varepsilon}$ . Denote  $\tilde{l}_0(\lambda) \equiv l_{\varepsilon}(\lambda^{-1})l_{\varepsilon}^{-1}(0)$ . In particular this means

$$\tilde{l}_0(-i\varepsilon) = l_{\varepsilon}(i\varepsilon^{-1})l_{\varepsilon}^{-1}(0) = b_{\varepsilon}g_{\varepsilon}^{-1}. \quad (60)$$

Finally, the comparison of Eq.(60) with Eq.(54) gives  $\tilde{l}_0(-i\varepsilon) = l_0(-i\varepsilon)$  which means  $\Xi_\varepsilon(g_\varepsilon) = g_0$ .

Although the proof is by now finished let us also mention that the independence of the left hand side of Eq.(57) on  $\varepsilon$  gives the  $\varepsilon$ -independence of the right hand side which in turn implies the  $\varepsilon$ -independence of the quantities  $u_\pm^\varepsilon$  of Eq.(22). Thus, as we have promised to show in Section 3.1, we indeed succeed to find for each  $\varepsilon$  a solution of the Yang-Baxter model  $g_\varepsilon$  (given by Eq.(53)) in such a way the quantities  $u_\pm^\varepsilon$  do not depend on  $\varepsilon$ .

#

Theorem 3 permits to translate into the Yang-Baxter context the solution generating techniques developped for the principal chiral model (cf. [12, 10, 3, 2, 7, 9]). It is just enough to dress a principal chiral solution and apply to the result the map  $\Xi_\varepsilon^{rec}$ .

## 4 Conclusions and outlook

This paper contains two principal results: the construction of the Lax pair (3) of the Yang-Baxter  $\sigma$ -model and the explicite description of the one-to-one map  $\Xi_\varepsilon^{rec}$  between the space of solutions of the principal chiral model and the space of solutions of the Yang-Baxter model. In particular, any solution generating method on the principal chiral side, like the dressing of solutions [12, 10, 3, 2, 7, 9], can be readily transferred into the Yang-Baxter context. It is just enough to dress a principal chiral solution and apply to the result the map  $\Xi_\varepsilon^{rec}$ .

We see three directions in which the results of this article could be developed further. One of them concern the T-duality story which is naturally associated to any Poisson-Lie symmetric  $\sigma$ -model [4]. In particular, the bi-Yang-Baxter model of Section 3.2, which we have introduced as the tool for deriving the Lax pair (3), deserves itself attention as the model Poisson-Lie T-dualizable from both right and left side. Combining the left and the right duality, a novel nontrivial dynamical equivalence of two  $\sigma$ -model living on the target of the group  $AN$  should be obtained.

The second direction concerns again the bi-Yang-Baxter model, but not from the point of view of the T-duality but rather that of integrability. We believe

that the work [8] of Sochen could be of some relevance for finding a hypothetical Lax pair of the bi-Yang-Baxter model. In fact, Sochen has formulated the problem of finding a Lax pair for a general  $\sigma$ -model on a group manifold which is neither left nor right symmetric in the ordinary sense of this word. He was able to write down certain overdetermined system of nonlinear equations for the coefficients of Lax matrices. As far as we know, there is no nontrivial solution of Sochen's equations available. It may be that for the special case of the bi-Yang-Baxter model such a solution could be found.

The third direction to develop is less concret than the two previous ones but it may have, perhaps, a broader range of applications. It concerns the important role which was played in this paper by the interpretation of the spectral parameter as the Poisson-Lie deformation parameter. This interpretation appeared first of all in Section 3.1 in comparison of the Poisson-Lie flat connection  $B^\varepsilon$  associated to the Yang-Baxter model with the Zakharov-Mikhailov Lax connection  $A^0(\lambda)$  associated to the principal chiral model. Secondly, and more directly, it appeared in the study of the extended solutions of the principal chiral model. Indeed, applying the Iwasawa map on the extended solution  $l_0(\lambda = -i\varepsilon)$ , we have obtained the ordinary solution of the Yang-Baxter  $\sigma$ -model which itself is the Poisson-Lie  $\varepsilon$ -deformation of the principal chiral model. We believe that the fact that the spectral parameter can be interpreted as the deformation parameter will serve as a fruitful insight in the study of dynamics of other integrable models than those studied in this article.

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